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Boundary Value Problems for Differential Equations in Banach Space

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1. INTRODUCTION

Consider the integral equation

$$u(t) = W(t, 0)\, u(0) + \int_0^t W(t, s) f(s, u(s))\, ds \qquad (t \in J) \tag{I}$$

in a Banach space X together with a linear boundary condition

$$Qu = a. \tag{II}$$

Here $J = [0, T]$ is an interval of real numbers, $\{W(t, s): 0 \leqslant s \leqslant t \leqslant T\}$ is a strongly continuous family of evolution operators on X, and $f: J \times X \to X$ is a function. Q in (II) is a continuous linear operator mapping $C(J, X)$ into X and $a \in X$ is given.

We offer sufficient conditions for the existence of a solution to the nonlinear integral equation (I) which also satisfies the boundary condition (II). We assume f satisfies Caratheodory-type conditions which do not imply that f maps bounded sets in $J \times X$ into bounded sets in X.

If the evolution system $W(t, s)$ has a closed densely defined family of generators $A(t)$ then equation (I) is an integrated form of the differential equation

$$u'(t) = A(t)\, u(t) + f(t, u(t)). \tag{I'}$$

The approach we take is motivated by the papers of Opial [8], Conti [2], Chow and Lasota [1], and Kartsatos [5] on boundary value problems for ordinary differential equations in R^n, and by the paper of Pazy [9] on initial value problems in Banach space.

In Section 2 we give notation and some preliminary results. In Section 3 we give our main results, and in Section 4 we apply these results to boundary problems for some parabolic partial differential equations.

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2. Preliminaries

Let R denote the set of real numbers and let $J = [0, T]$ be a compact interval of real numbers. Let X be a Banach space with norm $|\cdot|$. By $C = C(J, X)$ we mean the Banach space of continuous X valued functions defined on J with norm $\|\cdot\|$ defined by

$$\|u\| = \sup_{t\in J} |u(t)|$$

for $u \in C$. The space of bounded linear transformations on X will be denoted by $L(X)$, and norms in $L(X)$ by $|\cdot|$.

We assume the following for the evolution system $W(t, s)$:

$$W(t,s) \in L(X) \quad \text{whenever} \quad 0 \leqslant s \leqslant t \leqslant T \quad \text{and for each } x \in X \text{ the mapping } (t,s) \to W(t,s)\,x \text{ is continuous.} \tag{2.1}$$

$$W(t,s)\,W(s,r) = W(t,r) \quad \text{whenever} \quad 0 \leqslant r \leqslant s \leqslant t \leqslant T. \tag{2.2}$$

$$W(t,t) = I, \quad \text{the identity operator on } X, \text{ for each } t \in J. \tag{2.3}$$

$$W(t,s) \text{ is a compact operator whenever } t - s > 0 \quad (0 \leqslant s \leqslant t \leqslant T). \tag{2.4}$$

The reader is referred to [4] for sufficient conditions for (2.1)–(2.3) to hold for evolution systems with generator $A(t)$. If the conditions in [4, p. 108] are satisfied, and if for each $t \in J$ there is a number λ in the resolvent set of $A(t)$ such that the resolvent $R(\lambda; A(t))$ is compact then the generated evolution system will satisfy (2.4); see [3].

We will assume that $f: J \times X \to X$ satisfies the following.

(C1) For each $t \in J$ the function $f(t, \cdot): X \to X$ is continuous, and for each $x \in X$ the function $f(\cdot, x): J \to X$ is strongly measurable.

(C2) For every positive integer k there exists $g_k \in L^1(J)$ such that for a.a. $t \in J$

$$\sup_{|x| \leqslant k} |f(t,x)| \leqslant g_k(t).$$

For the definition and main results concerning strongly measurable functions, the reader is referred to [6, Sects. 3.5–3.8].

We need the following lemmas.

LEMMA 2.1. *Let* $\{W(t, s): 0 \leqslant s \leqslant t \leqslant T\}$ *satisfy* (2.1)–(2.4); *then for each fixed* $s \in [0, T)$ *the mapping* $t \to W(t, s)$ *from* $(s, T]$ *into* $L(X)$ *is continuous in the uniform operator topology on* $L(X)$. *Moreover, this continuity is uniform with respect to* s *in sets bounded away from* t, *i.e., as long as* $t - s \geqslant \beta$ *for any fixed* $\beta > 0$.

Proof. Let $0 < \beta < T$ be fixed and let $S = \{x \in X: |x| \leqslant M\}$ where $M = \sup\{|W(t, s)|: 0 \leqslant s \leqslant t \leqslant T\}$. M is finite by (2.1) and the principle of uniform boundedness. Fix t, $\beta < t \leqslant T$ and let $K = W(t, t - \beta) S$. The set K is totally bounded because $W(t, t-\beta)$ is compact. Thus given $\epsilon > 0$ there exists a finite subset $\{y_1, ..., y_n\}$ of S such that the n balls defined by $B_j = \{z \in X: |z - W(t, t-\beta) y_j| < \epsilon\}$ for $j = 1, 2, ..., n$, cover K.

Now choose $0 < \delta < 1$ so that $0 < h < \delta$ implies $|W(t+h, t-\beta) y_j - W(t, t-\beta) y_j| < \epsilon$ for $j = 1, 2, ..., n$. This is possible by the strong continuity of $W(t, s)$, (2.1).

Now let $x \in X$ with $|x| \leqslant 1$ and let $y = W(t - \beta, s) x$. Then $y \in S$ and $W(t, s) x = W(t, t-\beta) y$; hence there is an integer $j \in \{1, 2, ..., n\}$ such that $|W(t, t-\beta) y_j - W(t, t-\beta) y| < \epsilon$. We now have for $0 < h < \delta$:

$$\begin{aligned} |W(t+h, s) x - W(t, s) x| &= |W(t+h, t-\beta) y - W(t, t-\beta) y| \\ &\leqslant |W(t+h, t-\beta) y - W(t+h, t-\beta) y_j| \\ &\quad + |W(t+h, t-\beta) y_j - W(t, t-\beta) y_j| \\ &\quad + |W(t, t-\beta) y_j - W(t, t-\beta) y| \\ &\leqslant |W(t+h, t)| \epsilon + \epsilon + \epsilon \leqslant \epsilon(M+2). \end{aligned}$$

Since $\epsilon > 0$ was arbitrary and δ depended only on t and β, this proves the lemma for continuity from the right. The proof for continuity from the left requires little change and is omitted.

If $f: J \times X \to X$ satisfies (C1) and $u \in C(J, X)$ then $V: J \to X$ defined by $V(t) = f(t, u(t))$ is strongly measurable. The proof of this in case X is infinite dimensional parallels that of the case for $X = R^n$ as presented in [10, p. 92], making use of the results in [6], particularly Theorem 3.5.4, p. 74. If f satisfies (C2) in addition to (C1) then also V is Bochner integrable on J, and again the argument parallels that of [10, p. 92] for finite dimensional spaces, making use of the dominated convergence theorem for Bochner integrals [6, p. 83].

We therefore define an operator $\psi: C \to C$ by

$$\psi u(t) = \int_0^t W(t, s) f(s, u(s))\, ds \tag{2.5}$$

for each $u \in C$ and $t \in J$.

LEMMA 2.2. *Let ψ be defined by* (2.5). *Then ψ maps C into itself and is completely continuous, i.e., ψ is continuous and maps bounded sets in C into relatively compact sets.*

Proof. Let $B_k = \{u \in C: \| u \| \leqslant k\}$ for some $k \geqslant 1$. We first show that ψ maps B_k into an equi-continuous family and *a fortiori* $\psi u \in C$ for $u \in C$. Let $u \in B_k$ and $t, \tau \in J$. Let $\epsilon > 0$. Then if $0 < \epsilon < t < \tau$

$$\begin{aligned}| \psi u(t) - \psi u(\tau)| &\leqslant \left| \int_t^\tau W(\tau, s) f(s, u(s))\, ds \right| \\ &\quad + \left| \int_0^{t-\epsilon} [W(t, s) - W(\tau, s)] f(s, u(s))\, ds \right| \\ &\quad + \left| \int_{t-\epsilon}^t [W(t, s) - W(\tau, s)] f(s, u(s))\, ds \right| \\ &= I_1 + I_2 + I_3 .\end{aligned}$$

We have for estimates

$$I_1 \leqslant M \int_t^\tau g_k(s)\, ds$$

$$I_2 \leqslant \int_0^{t-\epsilon} | W(t, s) - W(\tau, s)|\, g_k(s)\, ds$$

$$I_3 \leqslant 2M \int_{t-\epsilon}^t g_k(s)\, ds.$$

The bounds on I_1 and I_3 may be made small by choosing t close to τ and ϵ sufficiently small. By Lemma 2.1 $W(t, s)$ is continuous in the operator norm, uniformly for $t - s \geqslant \epsilon$ and thus $I_2 \to 0$ as $t \to \tau$, or as $\tau \to t$. Thus ψ maps B_k into an equicontinuous family of functions. Clearly the family ψB_k is uniformly bounded also.

Now let $0 < t \leqslant T$ be fixed and ϵ a real number satisfying $0 < \epsilon < t$. Define $\psi_\epsilon(t, \cdot)$ by $\psi_\epsilon(t, u) = \int_0^{t-\epsilon} W(t, s) f(s, u(s))\, ds$ for all $u \in B_k$. By (2.2) $\psi_\epsilon(t, u) = W(t, t - \epsilon) \int_0^{t-\epsilon} W(t - \epsilon, s) f(s, u(s))\, ds$. By hypothesis $W(t, t - \epsilon)$ is a compact operator and hence the set $K_\epsilon(t) = \{\psi_\epsilon(t, u): u \in B_k\}$ is precompact in X. Also,

$$\begin{aligned}| \psi u(t) - \psi_\epsilon(t, u)| &\leqslant \int_{t-\epsilon}^t | W(t, s) f(s, u(s))|\, ds \\ &\leqslant M \int_{t-\epsilon}^t g_k(s)\, ds.\end{aligned}$$

Thus there are precompact sets arbitrarily close to the set $k_0(t) = \{\psi u(t): u \in B_k\}$ and therefore $k_0(t)$ is precompact. By the Arzela–Ascoli theorem it follows that ψ maps B_k into a precompact set in C.

It remains to show that $\psi: C \to C$ is continuous. Let $\{u_n\}_{n=1}^\infty \subseteq C$ with $u_n \to u$ in C. Then there is an integer N such that $| u_n(t)| \leqslant N$ for all n and $t \in J$, so that $u_n \in B_N$ and $u \in B_N$. By (C1) $f(t, u_n(t)) \to f(t, u(t))$ for each $t \in J$ and since $| f(t, u_n(t)) - f(t, u(t))| \leqslant 2g_N(t)$ we have by dominated convergence

$$\| \psi u_n - \psi u \| = \sup_{t \in J} \left| \int_0^t W(t, s) [f(s, u_n(s)) - f(s, u(s))] \, ds \right|$$
$$\leqslant M \int_0^T |f(s, u_n(s)) - f(s, u(s))| \, ds \to 0$$

as $n \to \infty$. Thus ψ is continuous. This completes the proof of the lemma.

The operator Q in (II) is a continuous linear mapping from C into X. We define $\tilde{Q} \in L(X)$ by

$$\tilde{Q}x = Q[W(\cdot, 0) \, x] \tag{2.6}$$

for all $x \in X$. We will assume in the sequel that $\tilde{Q}$ has a bounded inverse. We will need the following lemma, the proof of which is straightforward and is omitted.

Lemma 2.3. *Let* $\{W(t, s): 0 \leqslant s \leqslant t \leqslant T\}$ *satisfy* (2.1)–(2.3) *and let* f *satisfy* (C1) *and* (C2). *Assume* $\tilde{Q}$ *has a bounded inverse* $\tilde{Q}^{-1}$. *Then a function* $u \in C$ *solves* (I), (II) *if and only if* u *is a solution to the equation*

$$u = Vu + \psi u \tag{2.7}$$

where ψ *is defined by* (2.5) *and* $V: C \to C$ *is defined by*

$$Vu(t) = W(t, 0) \, \tilde{Q}^{-1}[a - Q\psi(\cdot, u)] \qquad \text{for } u \in C \quad \text{and} \quad t \in J. \tag{2.8}$$

Lemma 2.4. *Let* $N = V + \psi$. *Then* $N: C \to C$ *and is completely continuous.*

Proof. It is clear from the definition of V and by Lemma 2.2 that N maps C into itself. Moreover since the mapping $x \to W(\cdot, 0) \, x$ is bounded and linear from X into C the operator V is continuous and compact since for all $k > 0$ the set $\{\tilde{Q}^{-1}[a - Q\psi(\cdot, u)]: u \in B_k\}$ is precompact in X. Hence $N = V + \psi$ is continuous and compact.

3. Main Results

The following is the main result of this paper.

Theorem 3.1. *Let* $W(t, s)$ *be an evolution system satisfying* (2.1)–(2.4). *Assume* (C1) *and* (C2) *hold, and that* $\tilde{Q} \in L(X)$ *has a bounded inverse. Suppose also that*

$$\varliminf_{k \to \infty} k^{-1} \int_0^T g_k(s) \, ds = \alpha < +\infty. \tag{3.1}$$

Then there is a strongly continuous function u: $J \to X$ *which satisfies* (I) *and* (II) *provided*

$$(M \cdot |\tilde{Q}^{-1}| \cdot |Q| + 1)\, M\alpha < 1. \tag{3.2}$$

Proof. We have shown that problem (I), (II) is equivalent to the equation

$$u = Nu.$$

It will be shown that N has a fixed point. By Lemma 2.4 N: $C \to C$ is completely continuous. We will show that for some natural number n, $NB_n \subseteq B_n$. Then N has a fixed point in B_n by the Leray–Schauder theorem. Suppose to the contrary that for each natural number k there is a function $u_k \in B_k$ with $Nu_k \notin B_k$, i.e. with $\| Nu_k \| > k$. Then

$$1 < \frac{1}{k} \| Nu_k \|.$$

Thus we have $1 \leqslant \underline{\lim}_{k\to\infty} k^{-1} \| Nu_k \|$. But also

$$\begin{aligned}
\underline{\lim}_{k\to\infty} k^{-1} \| Nu_k \| &\leqslant \underline{\lim}_{k\to\infty} k^{-1}\{\sup_{t\in J} | Vu_k(t)| + \sup_{t\in J} | \psi u_k(t)|\} \\
&\leqslant \underline{\lim}_{k\to\infty} k^{-1} \left\{M\, |\tilde{Q}^{-1}| \left(|a| + |Q| \int_0^T Mg_k(s)\, ds\right) + \int_0^T Mg_k(s)\, ds\right\} \\
&= M\, |\tilde{Q}^{-1}| \cdot |Q| \cdot M \cdot \alpha + M\alpha \\
&= (M\, |\tilde{Q}^{-1}| \cdot |Q| + 1)\, M\alpha < 1,
\end{aligned}$$

a contradiction. Hence, $NB_n \subseteq B_n$ for some positive integer n, and by the Leray–Schauder theorem there is $u \in B_n$ with $u = Nu$. This completes the proof of the theorem.

Chow and Lasota [1] point out that solutions to boundary value problems for ordinary differential equations often exist if the boundary operator is close to the initial value operator. We have,

COROLLARY 3.1. *Assume* $W(t, s)$ *satisfies* (2.1)–(2.4) *and that* f *satisfies* (C1), (C2), *and* (3.1) *with* $\alpha = 0$. *Define* L: $C \to X$ *by* $Lu = u(0)$, *and let* Q: $C \to X$ *be a continuous linear mapping. If* $|Q - L| < M^{-1}$ *then* (I), (II), *has a solution.*

Proof. All we need show is that $\tilde{Q}$ has a bounded inverse. Note that $\tilde{L} = I$, the identity on X. Thus

$$\begin{aligned}
|\tilde{Q} - I| &= \sup_{|x|=1} |QW(\cdot, 0)\, x - LW(\cdot, 0)\, x| \\
&\leqslant |Q - L| \sup_{|x|=1} \| W(\cdot, 0)\, x \| \\
&\leqslant |Q - L| \cdot M < 1.
\end{aligned}$$

Thus $\tilde{Q}$ has a bounded inverse.

COROLLARY 3.2. *Let $W(t, s)$ satisfy* (2.1)–(2.4) *and let f satisfy* (C1) *and* (C2). *Then for each $u_0 \in X$ there is a number $t_1 = t_1(u_0) > 0$ and a function $u \in C([0, t_1], X)$ which solves* (I) *on* $[0, t_1]$, *and such that* $u(0) = u_0$.

Proof. Choose $\rho > 0$ and let $B_\rho(u_0) = \{x \in X: | x - u_0 | \leqslant \rho\}$. Define a function $\tilde{f} : J \times X \to X$ by

$$\begin{aligned} \tilde{f}(t, x) &= f(t, x) \quad \text{if} \quad 0 \leqslant t \leqslant T \quad \text{and} \quad x \in B_\rho(u_0) \\ &= f(t, \tilde{x}) \quad \text{if} \quad 0 \leqslant t \leqslant T \quad \text{and} \quad x \notin B_\rho(u_0) \end{aligned}$$

where $\tilde{x} = u_0 + \rho \cdot (u_0 - x) \cdot | u_0 - x |^{-1}$. Let N be an integer such that $B_\rho(u_0) \subseteq B_N$. Then $|\tilde{f}(t, x)| \leqslant g_N(t)$ for all $x \in X$ and a.e. $t \in J$.

The hypotheses of Theorem 3.1 are thus satisfied for $\tilde{f}$ with $Qu = u(0)$ and $\alpha = 0$. Hence there is a function $u \in C(J, X)$ with $u(0) = u_0$ satisfying (I) with $\tilde{f}$ in place of f. However, u is continuous so there exists $t_1 > 0$ such that $| u(t) - u_0 | < \rho$ for $0 \leqslant t \leqslant t_1$. Since $f(t, x) = \tilde{f}(t, x)$ on $[0, t_1] \times B_\rho(u_0)$, it follows that $u(t)$ satisfies (I) on $0 \leqslant t \leqslant t_1$.

The following result concerns periodic solutions.

COROLLARY 3.3. *Let $\{W(t, s): 0 \leqslant s \leqslant t < \infty\}$ satisfy* (2.1)–(2.4) *for all $T > 0$ and suppose*

(i) *There is a number $p > 0$ such that $W(t + p, s + p) = W(t, s)$ for $0 \leqslant s \leqslant t < \infty$.*

(ii) *$W(p, 0)\, x = x$ iff $x = 0$.*

(iii) *$f: [0, \infty) \times X \to X$ satisfies* (C1) *for all $T > 0$ and $f(t + p, x) = f(t, x)$ for all $t \geqslant 0$ and $x \in X$.*

(iv) *For each positive integer k there is a function $g_k \in L^1(0, p)$ with*

$$\sup_{|x| \leqslant k} |f(t, x)| \leqslant g_k(t) \qquad \text{for a.e. } t \in [0, p],$$

and

$$\varliminf_{x \to \infty} k^{-1} \int_0^p g_k(s)\, ds = 0.$$

Then there exists a p-periodic solution to the integral equation (I), *valid for all* $t \geqslant 0$.

Proof. We first prove the existence of a solution to the boundary problems

$$z(t) = W(t, 0)\, z_0 + \int_0^t W(t, s) f(s, z(s))\, ds \qquad (0 \leqslant t \leqslant p) \tag{3.3}$$

$$Qz = z(0) - z(p) \equiv 0 \tag{3.4}$$

and then show that the solution to this problem may be extended p-periodically to all of $[0, \infty)$ as a solution of (I).

Here $\tilde{Q}$ is defined by $\tilde{Q} = I - W(p, 0)$. Thus $\tilde{Q}$ is a compact perturbation of the identity. By (ii) and the Fredholm alternative $\tilde{Q}$ has a bounded inverse. It is now a consequence of the remaining hypotheses together with Theorem 3.1 that there exists a solution $z(t)$ to (3.3), (3.4). If we now define $u(t)$ on $[0, 2p]$ by $u(t) = z(t)$ if $0 \leqslant t \leqslant p$ and $u(t) = z(t - p)$ if $p < t \leqslant 2p$ an application of (i), (iii), and a change of variables will show that $u(t)$ is a solution to the integral equation (I) on $[0, 2p]$ and satisfies $u(t + p) = u(t)$ for $0 \leqslant t \leqslant p$. Continuing in this manner the solution may be extended p-periodically to all of $[0, \infty)$.

4. An Example

We wish to illustrate the results of the previous section by showing their applicability to a parabolic partial differential equation with a nonlinearity satisfying Caratheodory conditions.

Consider the following differential equation with boundary and periodicity conditions.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial y^2} + f(t, y, u(t, y)) \qquad (t > 0, 0 < y < 1).$$

$$u(t, 0) = u(t, 1) = 0 \qquad (t > 0)$$

$$u(t + w, y) = u(t, y) \qquad \text{for all } t \geqslant 0 \quad \text{and} \quad y \in [0, 1].$$

Here $f: R^+ \times [0, 1] \times R \to R$ is a function and w is a fixed positive number. We assume that

(a) $f(t + w, y, u) = f(t, y, u)$ for all $(t, y, u) \in R^+ \times [0, 1] \times R$.

(b) For each fixed $t \in R^+$, $f(t, y, u)$ is a continuous function of (y, u).

(c) For each fixed $(y, u) \in [0, 1] \times R$, $f(t, y, u)$ is a (Lebesgue) measurable function of t.

(d) There is a number β, $0 < \beta < 1$, and a function $g \in L^1(0, w)$ such that $f(t, y, u)$ satisfies the inequality

$$|f(t, y, u)| \leqslant g(t)(|u|^\beta + C)$$

for all $(t, y, u) \in [0, p] \times [0, 1] \times R$.

Under these conditions an abstract formulation of (4.1) has a mild solution.

Let $X = L^2(0, 1)$ and let $D \subseteq X$ be the set

$$D = \{u \in X : u, u' \in AC, u'' \in X \text{ and } u(0) = u(1) = 0\}.$$

(Here AC denotes the functions absolutely continuous on $[0, 1]$.)

Define $A: D \to X$ by $Au = u''$. Define a Nemytskii operator $F: R^+ \times X \to X$ by the rule $F(t, u)(y) = f(t, y, u(y))$ for all $(t, u) \in R^+ \times X$ and $y \in [0, 1]$. It is known that for each $t \in R^+$, $F(t, \cdot)$ maps $X = L^2(0, 1)$ into itself continuously (see, e.g., [7, p. 161]). Problem (4.1) may now be written in the abstract form

$$\begin{aligned} u'(t) &= Au(t) + F(t, u(t)) \qquad (t > 0) \\ u(t + w) &= u(t). \end{aligned} \tag{4 2}$$

It is known [7, p. 309] that A generates a strongly continuous semigroup $T(t)$ on X and $T(t)$ is compact for each $t > 0$. Moreoevr, there are numbers $\delta > 0$ and $M \geqslant 1$ such that $|T(t)| \leqslant Me^{-\delta t}$ for all $t \geqslant 0$. It follows that the only periodic solution to the equation $u'(t) = Au(t)$ is the trivial solution $u(t) = 0$. Hence the only solution to the equation $T(w)x = x$ is $x = 0$. Letting $W(t, s) = T(t - s)$, $W(t, s)$ thus satisfies (i) and (ii) of Corollary 3.3.

We must show that the operator F satisfies (iii) and (iv) of Corollary 3.3. Let $u \in X$ be fixed. We must show that $F(\cdot, u)$ is strongly measurable. Since $X = L^2(0, 1)$ is separable it suffices to show that $F(\cdot, u)$ is weakly measurable [6, p. 73]. Let $\phi \in X^* = L^2(0, 1)$. Then $f(t, y, u(y))\,\phi(y)$ is measurable on $[0, w] \times [0, 1]$ and by condition (d) we may show

$$\iint_E |f(t, y, u(y))\,\phi(y)|\,dy \times dt < +\infty, \qquad [0, w] \times [0, 1] = E.$$

Hence $f(t, y, u(y))\,\phi(y)$ is integrable on $[0, w] \times [0, 1]$ and by Fubini's theorem

$$\langle F(t, u), \phi \rangle = \int_0^1 f(t, y, u(y))\,\phi(y)\,dy$$

is a Lebesgue measurable function of t. Hence $F(t, u)$ is weakly measurable in t for each $u \in X$, and is thereby also strongly measurable in t for each $u \in X$.

The hypothesis that the number β satisfies $0 < \beta < 1$ may now be used to show that F satisfies (iv) of Corollary 3.3. We omit the argument. Corollary 3.3 now implies that there is a mild solution to (4.2), that is, a solution to the integral equation

$$u(t) = T(t)u(0) + \int_0^t T(t - s)\,F(s, u(s))\,ds, \qquad t > 0$$

satisfying

$$u(t + w) = u(t), \qquad \text{for all } t \geqslant 0.$$

References

1. S.-N. Chow and A. Lasota, On boundary value problems for ordinary differential equations, *J. Differential Equations* **14** (1973), 326–337.
2. R. Conti, Recent trends in the theory of boundary value problems for ordinary differential equations, *Boll. Un. Mat. Ital.* (3) **22** (1967), 135–178.
3. W. Fitzgibbon, Semilinear functional differential equations in banach space, *J. Differential Equations* **29** (1978), 1–14.
4. A. Friedman, "Partial Differential Equations," Holt, Rinehart & Winston, New York, 1969.
5. A. G. Kartsatos, A boundary value problems on an infinite interval," *Proc. Edinburgh Math. Soc.*, in press.
6. E. Hille and R. Phillips, "Functional Analysis and Semigroups," Colloq. Publ. 31, Amer. Math. Soc., Providence, R.I., 1957.
7. R. H. Martin, Jr., "Nonlinear Operators and Differential Equations in Banach Spaces," Wiley, New York, 1976.
8. Z. Opial, Linear problems for systems of nonlinear differential equations, *J. Differential Equations* **3** (1967), 580–594.
9. A. Pazy, A class of semi-linear equations of evolution, *Israel J. Math.* **20** (1975), 23–36.
10. W. T. Reid, "Ordinary Differential Equations," Wiley, New York, 1971.